On ergodicity of highly degenerate hybrid stochastic control systems

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Abstract

For switching diffusion systems with a homogeneous closed-loop control we establish existence and uniqueness of solution of the equation, and rate of convergence to the stationary regime.

1 Introduction

This work stems from the investigation of F. Campillo and E. Pardoux into the issue of a vehicle suspension device, see [7, 8]. It is also an extension of recent works [2] and [1].

Stochastic ergodic control – in particular, with expected average in time with infinite horizon as cost functional – proved to be a useful tool for constructing a closed-loop control of a vehicle suspension device, cf. [9] and references therein. On theory of ergodic control for diffusion processes see, e.g., [4]. In [2] we have generalized the model of the suspension device to a multi-regime one. That is, several types of the road surfaces were admitted and it

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was assumed that the type of the road surface determines a gear box regime and, hence, also a working regime of the suspension device. This object may be described by a hybrid system (cf. [6]) with dynamics of a switching diffusion: position of the device is X, its velocity Y, and the type of the road V. Switchings constitute a *conditional* Markov chain (see [16, 10, 15]), with intensities conditional on the state of the diffusion component X_t. The novelty in comparison to the latter works is degenerate diffusion and discontinuous coefficients (as in [2]). Similar equations without switching have been considered in [1]. In comparison to [2], we consider below more general systems, which, in particular, allow higher degeneracy – the latter admits some natural physical interpretation – and dependence of switching process intensities of the continuous component.

The essential issue in ergodic control is ergodicity of the controlled process in the sense of Markov processes (see [13], [12, Ch.6.3]). It will be shown that under every homogeneous admissible control policy the distribution of the process converges in the long run to its limit at exponential rate uniformly over all admissible control policies and locally uniform with respect to initial conditions. A similar result holds true for mixing rate, however, we postpone the details until further papers.

We want to extend our previous results in three directions: 1) to cover more general multi-dimensional hamiltonian systems, including stochastic differential equations of Langevin – Smoluchowsky type; 2) to allow switching intensities dependent on diffusion state component; 3) to show convergence joint for the couple (X, V) (in our earlier paper intensities of V did not depend on X and this question did not arise). For simplicity we restict our presentation to the case of R^3; the study of the general situation is postponed till further papers. Notice that the case of R^2 was considered in [2], so that R^3 is the "next simplest" option.

Hence, let us start with an SDE system in R^3 without switching,

$$\begin{array}{rcll} dX_t^1 & = & X_t^2\,dt, & X_0^1 = x^1 \in R, \\ dX_t^2 & = & X_t^3\,dt, & X_0^2 = x^2 \in R, \\ dX_t^3 & = & b(t, X_t^1, X_t^2, X_t^3)\,dt + dW_t, & X_0^3 = x^3 \in R. \end{array}$$

Here W is a standard Wiener process. Due to the control origin of the setting, no regularity is assumed on the drift term b, which is just Borel measurable of not more than linear growth.

Now let us introduce switching. Let $(V_t,\, t \ge 0)$ be a conditional Markov chain in continuous time given "frozen" value of $X_t = x$ and taking values in a finite set $S = \{1, 2, \dots, N\}$ with a generator $Q(t, x) = (q_{ij}(t, x))_{N\times N}, x \in R^3$.

This matrix determines transition probabilities over a small period of time given $X_t = x$,

$$\mathbf{P}(V_{t+\Delta} = j | V_t = i, X_t = x) = \begin{cases} q_{ij}(t,x)\Delta + o(\Delta) \text{ if } i \neq j, \\ 1 + q_{ii}(t,x)\Delta + o(\Delta) \text{ if } i = j, \end{cases} \tag{1}$$

where $\Delta \downarrow 0$. For $j = i$ the value q_{ii} is defined as $q_{ii} = -\sum_{j:\, j\neq i} q_{ij}$. Now consider a hybrid SDE system in $R^3 \times S$,

$$\begin{array}{rcll} dX^1_t &=& X^2_t\, dt, & X^1_0 = x^1 \in R, \\ dX^2_t &=& X^3_t\, dt, & X^2_0 = x^2 \in R, \\ dX^3_t &=& b(t, X^1_t, X^2_t, X^3_t, V_t)\, dt + dW_t, & X^3_0 = x^3 \in R. \end{array} \tag{2}$$

Here W is a standard Wiener process, and the drift term b is a Borel measurable function. In order to define the process (X_t, V_t) rigorously we should present its two component generator G_t at $t \geq 0$. The latter acts on a function $f(x,v)$, $x \in R^3$, $v \in S$ according to the rule,

$$\begin{aligned} (G_t f)(x,v) &= \sum_j q_{vj}(t,x) f(x,j) \\ &+ x^2 f_{x^1}(x,v) + x^3 f_{x^2}(x,v) + \frac{1}{2} f_{x^3x^3}(x,v) + b(t,x,v) f_{x^3}(x,v). \end{aligned} \tag{3}$$

As was prompted above, we always assume uniform linear growth assumption: for some constant $C > 0$

$$\sup_{v\in S} |b(t,x,v)| \leq C(1+|x|). \tag{4}$$

As for intensities q_{ij}, we assume them to be contiuous functions of x, uniformly bounded and uniformly positive for each t,

$$q_{ij}(t,\cdot) \in C(R^3), \qquad 0 < \inf_{i\neq j;x,t} q_{ij}(t,x) \leq \sup_{i\neq j;x,t} q_{ij}(t,x) < \infty. \tag{5}$$

Conditions (4) and (5) are the only assumption needed for existence theorem, while continuity has been added only for simplicity of presentation; it is likely that non-degeneracy of inensities may be also dropped. In the theorem about Lyapunov function we assume homogeneity in time and a special structure of the drift b similar to that in [8]. Namely, $q_{ij} = q_{ij}(x)$ and the drift in that theorem *will not depend on* t and will have a form,

$$\begin{gathered} b(x^1,x^2,x^3,v) = -u(x^1,x^2,x^3,v)x^3 - \beta^1 x^1 - \beta^2 x^2 + \tilde{b}(x^1,x^2,x^3,v), \\ \beta^1 > 0, \quad \beta^2 > 0, \quad 0 < u_1 \leq u(\cdot) \leq u_2 < \infty, \quad \tilde{b}(\cdot) \text{ bounded}. \end{gathered} \tag{6}$$

For simplicity, in this presentation $\tilde{b} \equiv 0$. The main result is Theorem 4 below, however, all Theorems 1–3 are important on their own.

2 Results

2.1 Weak existence and uniqueness

Theorem 1 *Let the system (1) – (2) with generator (3) satisfy the conditions (4) and (5). Then this system has a unique in distribution weak solution* (X, V) *on* $[0, \infty)$*, which is a strong Markov process. All trajectories of* V *are right-continuous step functions with no accumulations of jumps.*

Proof of Theorem 1 uses Peano discretization and tightness of measures. For discretized system the method from [1] and its extension proposed in [14] is used; in turn they extend the existence result from [3]. Strong Markov propery follows from weak uniqueness as in [11].

2.2 Lyapunov function

Stability or recurrence of the system follows from Lyapunov function; the latter may be constructed following an extension of the trick from [7]. Let us denote

$$\tau_R := \inf(s \geq 0 : \ |X_s| \leq R).$$

Theorem 2 *Let the system (1) – (2) with generator (3) satisfy the conditions (4), (5) and (6), with intensities* q *homogeneous in time. Then for* R *large enough, there exist* $C, \alpha > 0$ *such that for all* $t \geq 0$ *and any* x, v*,*

$$\sup_{0 \leq t} \mathbf{E}_x |X_t|^2 1(t \leq \tau_R) \leq C(1 + |x|^2)$$

and

$$\mathbf{E}_{x,v} \exp(\alpha \tau_R) \leq C(1 + |x|^2).$$

Proof follows from a Lyapunov function

$$f(x^1, x^2, x^3) := \epsilon(x^1, x^2) B (x^1, x^2)^T + |x^3|^2 + \epsilon(c_1 xz + c_2 yz),$$

with $\epsilon > 0$ small enough,

$$B = \begin{pmatrix} b_{11} & \beta^1 \\ \beta^1 & \beta^2 \end{pmatrix}$$

and

$$b_{11} > \frac{|\beta^1|^2}{\beta^2}.$$

The latter condition ensures that the matrix B is positive definite for small values of $\epsilon > 0$. Lyapunov property is guaranteed by further conditions,

$$c_1\beta^1 > 0 \quad c_2\beta^2 > 2\beta^1,$$

and

$$(-c_1\beta^1)\,(2\beta^1 - c_2\beta^2) - (b_{11} - \frac{1}{2}c_1\beta^2 - \frac{1}{2}c_2\beta^1)^2 < 0.$$

It is easily verified that the family of such matrices B is non-empty.

2.3 Local Markov–Dobrushin's mixing

The next result is crucial for the method used in this paper. Denote $Z = (X, V)$, $B_R := \{(x, v) : \ |x| \le R\}$ and let $\mu_{s,s+T}(z_0, dz)$ be the transition measure for the process Z from s to $s+T$, and $\mu^{R'}_{s,s+T}(z_0, dz)$ – its restriction on trajectories where the component X does not attain the level R'. We are going to verify that for any R large enough there exist $R' \ge R$ and a suitable $T > 0$ such that

$$\inf_s \inf_{z_0, z_0' \in B_R} \int_{B'_R} \left(\frac{\mu^{R'}_{s,s+T}(z_0, dz)}{\mu^{R'}_{s,s+T}(z_0', dz)} \wedge 1 \right) \mu^{R'}_{s,s+T}(z_0', dz) > 0. \tag{7}$$

The density of one measure with respect to another is interpreted in the usual way, that is, as a density of the absolute continuous component. It suffices to establish a "conditional" version of (7), for z_0 and z_0' with equal second components given that these second components both stay at one fixed state,

$$\inf_{\substack{z_0, z_0' \in B_R: \\ v_0 = v_0' = \bar v \in S}} \int_{B'_R} \left(\frac{\mu^{R'}_{s,s+T}(z_0, dz \mid v = \bar v)}{\mu^{R'}_{s,s+T}(z_0', dz \mid v' = \bar v)} \wedge 1 \right) \mu^{R'}_{s,s+T}(z_0', dz \mid v' = \bar v) > 0. \tag{8}$$

Here $\cdot \mid v = \bar v$ (respectively, $\cdot \mid v' = \bar v$) signifies a conditional probability given that on the whole interval $[s, s+T]$, the process V_t (respectively, V_t') stays at $\bar v$.

Theorem 3 *Under the assumptions of Theorem 1, for any $R > 0$ there exist $R' \ge R$ and $T > 0$ such that the inequality (8) holds true.*

2.4 Convergence rate

Let $\mu_t^{x,v}$ denote the distribution of the pair (X_t, V_t).

Theorem 4 *Under the assumptions of the Theorem 2, there exists a unique stationary distribution μ_∞ on $\mathcal{B}(R^3) \times 2^S$ and there exist $C, c > 0$ such that*

$$\|\mu_t^{x,v} - \mu_\infty\|_{TV} \le C \exp(-ct)(1 + |x|^2), \quad t \in [0, \infty).$$

The Theorems 1 – 3 being established, the *proof* of the Theorem 4 follows in the same way as in [2]. The idea of tackling additional discrete component is as follows. Because all intensities are uniformly bounded and positive, discrete components of two versions of the process meet at some state in a unit of time with a positive probability and then both remain in that state during another unit of time. Continuous components should be already close and belong to some bounded set, which is achievable due to the Lyapunov function. Then during that second unit of time there is a positive conditional probability that continuous components could be coupled. More details about the method can be found in [5].

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